On the structure of local branchings

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Abstract

Local $n$-branchings play a central role in higher dimensional rewriting. In this paper we study the structure of these branchings. We define what a presentation of an $O$-algebra by a rewriting system is, for a general (non-symmetric, set-theoretic) operad $O$. In this setting, we define what a local $n$-branching is, and show that they form an $O$-algebra in the category of augmented symmetric simplicial sets, equipped with the appropriate monoidal structure. In particular when $O$ is the monoid operad, local $n$-branchings form a monoid in augmented symmetric simplicial sets.

Finally we define the notions of aspherical, Peiffer, overlapping and critical branchings in this setting, and recover in good cases some properties expected from the traditional monoid example.

1 Introduction

One application of higher dimensional rewriting is the construction of resolutions of objects presented by generators and relations. The first example of such constructions were applied to algebras (see [1]). More recently came applications to monoids [6] and PROs [5]. While there exist methods to produce standard resolutions in a functorial way, the resolutions thus constructed are very far from being minimal. On the other hand, minimal resolutions (when they exist) are often very hard to compute. Higher dimensional rewriting uses the combinatorial information given by a presentation to build a resolution that is both small and easy to compute.

Generators of such a resolution are usually indexed by critical branchings [6] (or variants thereof, such as Anick’s chains [1] for example). Those form a subfamily of the more general local branchings. Let us take the example of monoids. A word rewriting system is given by a set $E$ of "generators", and a set $R$ of "relations", together with applications $s, t : R \to E^*$, where $E^*$ denotes the free monoid generated by $E$. An example of rewriting system is often depicted as follows:

$$\langle a, s, t | f : st \to a, g : as \to ta \rangle$$

Here, $E = \{a, s, t\}$ and $R = \{f, g\}$, with applications defined by $s(f) = st, s(g) = as, t(f) = a$ and $t(g) = ta$.

A rewriting step is a triple $u fv$, with $u, v \in E^*$ and $f \in R$. Its source is $us(f)v$, and its target is $ut(f)v$. Finally an $n$-local branching is an $n$-tuple of rewriting steps sharing the same source. Such branchings are usually separated in three families:

- The branching $(fas, fas)$ of source $stas$, and more generally branchings where one rewriting step appears multiple times, is called an *aspherical branchings*

- The branching $(fas, stg)$ of source $stas$, and more generally branchings where rewriting steps apply to disjoint parts of the source, is called a *Peiffer branching*

- Other types of branchings, such as $(af, gt)$ of source $ast$ or $(afa, gta)$ of source $asta$ are called *overlapping branchings*.

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Finally there exists an order relation between overlapping branchings, corresponding to adding superfluous context to a branching. For example, the branching \((af, gt)\) is smaller than \((afa, gta)\). Overlapping branchings that are minimal for this ordering are called critical branchings.

In this paper, we define local branchings in a more general setting. Monoids are replaced by \(\mathcal{O}\)-algebras, where \(\mathcal{O}\) is a colored (set-theoretic, non-symmetric) operad, also called a multicategory. Such operads go back to Boardman and Vogt \cite{boardman1978little}. In Section 2, we introduce the notion of \(\mathcal{O}\)-rewriting system. Then in Section 3 to define the notion of local branching in this setting. And explore their structure, leading to the following Proposition:

**Proposition 3.6.** Let \(\mathcal{O}\) be an operad, and \((E, R)\) be an \(\mathcal{O}\)-rewriting system. The family of local branchings \(\mathcal{B}r(E, R)\) equipped with the applications \(\tilde{c}_i, \epsilon_i\), and \(\mathfrak{m}_{\mathcal{B}r}\) forms a simplicial \(\mathcal{O}\)-algebra, that is an \(\mathcal{O}\)-algebra in augmented symmetric simplicial sets.

In Section 4, we define the notions of aspherical, Peiffer, overlapping and critical branchings. Finally there exists an order relation between overlapping branchings, corresponding to adding superfluous context to a branching. For example, the branching \((f, g)\) is smaller than \((afa, gta)\). Overlapping branchings that are minimal for this ordering are called critical branchings.

**Example 2.3.** Many structures can be described as algebras over operads, and in particular:

- For any monoid \(M\), the operad of sets equipped with an action of \(M\). This operad is defined by \(\mathcal{O}(n) = \emptyset\) if \(n \neq 1\), and \(\mathcal{O}(1) = M\), where the composition induced by that of \(M\).

**Definition 2.6.** Let \(\mathcal{O}\) be an operad and let \(E\) be a set. Then the free \(\mathcal{O}\)-algebra generated by \(E\), denoted \(E^*\), is defined by:

\[
E^* = \bigcup_{n \in \mathbb{N}} \mathcal{O}(n) \times E^n
\]

Composition is defined in the straightforward way, using the composition in \(\mathcal{O}\).
Example 2.7. If $O$ is the monoid operad, and $E$ is a set, then $E^* = \cup_{n \in \mathbb{N}} E^n$ is the free monoid generated by $E$. If $O$ is the operad of sets equipped with an action of a monoid $M$, then $E^* = M \times E$, where $M$ acts on itself by composition on the left.

Definition 2.8. Let $O$ be an operad and $A$ be an $O$-algebra, and $R$ be a set. Then there is a free $A$-module generated by the sets $R$, that we denote by $R^+$. If $A$ is itself a free $O$-algebra generated by a set $E$, then the free $A$-module generated by the set $R$ can be described by:

$$R^+ = \bigsqcup_{n \in \mathbb{N}} \bigcup_{1 \leq i \leq n} O(n) \times E^{i-1} \times R \times E^{n-i}$$

Definition 2.9. An $O$-rewriting system is the data of two sets $E$ and $R$, and applications $s, t : R \to E^*$. The $O$-algebra presented by such a rewriting system is the $O$-algebra $A$ which is initial among the $O$-algebras $B$ such that there exists a morphism (of $O$-algebras) $f : E^* \to B$ such that $f \circ s = f \circ t$.

3 Local Branchings

Definition 3.1. Let $\Delta$ be the category of finite cardinals (with all maps between them). A presheaf over $\Delta$ is called an augmented symmetric simplicial set (originally defined by Grandis, see [3]). The category $\Delta$ is moreover equipped with a monoidal structure (induced by the sum of cardinals). By Day convolution [3], this induces a monoidal structure on augmented symmetric simplicial sets.

We call an $O$-algebra in augmented symmetric simplicial sets (using the monoidal structure previously defined) a simplicial $O$-algebra.

Example 3.2. Let us make explicit the notion of simplicial $O$-algebra, where $O$ is an operad. A simplicial $O$-algebra is given by a family of sets $S_n$ for $n \geq 0$ together with applications $\hat{\epsilon}_i : S_{n+1} \to S_n$ for $1 \leq i \leq n$ and $\epsilon_i : S_n \to S_{n+1}$ for $1 \leq i \leq n + 1$, making $S$ into a simplicial set. Moreover on each $n \geq 0$ there is an action of $\Sigma_n$ on $S_n$, which acts on the $\hat{\epsilon}_i$ and $\epsilon_i$ by acting on the index.

Finally, for any $m \in O(n)$, and any $i_1, \ldots, i_n$ in $\mathbb{N}$, an application

$$m_S : S_{i_1} \times \ldots \times S_{i_n} \to S_{i_1 + \ldots + i_n}$$

satisfying the appropriate conditions. In particular for example:

$$m_S(x_1, \ldots, x_{j-1}, \hat{\epsilon}_k x_j, x_{j+1}, \ldots, x_n) = \hat{\epsilon}_{i_1 + \ldots + i_{j-1} + k} m_S(x_1, \ldots, x_n)$$

Definition 3.3. Let $(E, R)$ be an $O$-rewriting system. A rewriting step $f$ is an element of $R^+$. We call $s(f)$ its source. A local $n$-branching (for $n > 0$) is an $n$-tuple $(f_1, \ldots, f_n)$ of rewriting steps of same source. We denote by $\text{Br}(E, R)_n$ the set of all $n$-local branchings. We extend that to $n = 0$ by saying that a $0$-local branching is just an element of $E^*$.

Definition 3.4. Let $(E, R)$ be an $O$-rewriting system. We define:

- For all $(f_1, \ldots, f_n) \in \text{Br}(E, R)_n$, and $1 \leq i \leq n$, let $\hat{\epsilon}_i(f_1, \ldots, f_n)$ be the branching

$$(f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_n)$$

- For all $(f_1, \ldots, f_n) \in \text{Br}(E, R)_n$, and $1 \leq i \leq n$, let $\epsilon_i(f_1, \ldots, f_n)$ be the branching

$$(f_1, \ldots, f_i, f_{i+1}, \ldots, f_n)$$

- For all $(f_1^1, \ldots, f_n^1) \in \text{Br}(E, R)_{i_1}, \ldots, (f_1^n, \ldots, f_n^n) \in \text{Br}(E, R)_{i_n}$ respectively of source $u_1, \ldots, u_n$ and $m \in O(n)$, let $m_{\text{Br}}(f_1^1, \ldots, f_n^n)$ be the branching:

$$(m(f_1^1, u_2, \ldots, u_n), \ldots, m(f_{i_1}^1, u_2, \ldots, u_n), \ldots, m(u_1, \ldots, u_{n-1}, f_{i_n}^n), \ldots, m(u_1, \ldots, u_{n-1}, f_n^n))$$

Finally, $\Sigma_n$ acts on $\text{Br}(E, R)$ by permuting the rewriting steps.
Example 3.5. In the case where \( \mathcal{O} \) is the monoid operad, let us look at the \( \mathcal{O} \)-rewriting system \((E, R)\) where \( E \) is the the set \{a, b, c\} and \( R = \{f, g\} \) with the source and target operations given by \( s(f) = ab \), \( t(f) = bc \), \( s(g) = ba \) and \( t(g) = aa \).

Then \((fa, ag)\) is a local 2-branching of source \( aba \). We can also form the product of \( f \) and \( g \) (seen as local 1-branchings): \( f \otimes g = (fba, abg) \), of source \( abba \). Multiplying the other way around gives \( g \otimes f = (gab, ba f) \), of source \( baab \).

Let \( \tau \) be the non-identity element of \( \Sigma_2 \). Note that applying \( \tau \) to \( f \otimes g \) does not yield \( g \otimes f \). Instead we have:

\[
\begin{align*}
f \otimes g &= (fba, abg) & g \otimes f &= (gab, ba f) \\
\tau \cdot (f \otimes g) &= (abg, fba) & \tau \cdot (g \otimes f) &= (ba f, gab)
\end{align*}
\]

The following proposition is a straightforward verification of the axioms.

Proposition 3.6. Let \( \mathcal{O} \) be an operad, and \((E, R)\) be an \( \mathcal{O} \)-rewriting system. The family of local branchings \( Br(E, R) \) equipped with the applications \( \hat{c}_i \), \( \epsilon_i \) and \( m_{Br} \) forms a simplicial \( \mathcal{O} \)-algebra, that is an \( \mathcal{O} \)-algebra in augmented symmetric simplicial sets.

## 4 Critical branchings

Using this structure, we can redefine the usual classification of local \( n \)-branchings

Definition 4.1. Let \( f_\bullet = (f_1, \ldots, f_n) \) be a local \( n \)-branching. If \( f_\bullet \) is (up to action of \( \Sigma_n \)) of the form \( \epsilon_i g \) then \( f \) is called an aspherical branching. If \( f_\bullet \) is (up to action of \( \Sigma_n \)) of the form \( m(g_1, \ldots, g_n) \) where \( g_1 \) is an \( n_1 \)-branching with \( n_1 > 0 \), then \( f_\bullet \) is called a Peiffer branching. Otherwise, then \( f_\bullet \) is called a overlapping branching.

We define a relation \( \preceq \) on local branchings by saying, for all local branchings \( f_\bullet \), any \( x_1, \ldots, x_n \in E \), any \( m \in \mathcal{O} \) and any \( 1 \leq i \leq n \): \( f_\bullet \preceq m(x_1, \ldots, x_{i-1}, f_\bullet, x_i, \ldots, x_n) \).

Lemma 4.2. The relation \( \preceq \) on local branchings is a pre-order, that is it is reflexive and transitive.

Example 4.3. Note that the relation \( \preceq \) is not necessarily anti-symmetric. For example, let \( \mathcal{O} \) be the operad of sets equipped with an action of \( \Sigma_2 \). Let \( E = \{a, b\} \) and \( R = \{f\} \), with \( s(f) = a \) and \( t(f) = b \). This defines an \( \mathcal{O} \)-rewriting system. Moreover, let \( m \) be the non-trivial element of \( \mathcal{O}(1) \). Then we have \( f \preceq m(f) \preceq m(m(f)) = f \), where \( m(f) \) is a rewriting step of source \( m(a) \).

Definition 4.4. A local \( n \)-branching is a critical branching if it is an overlapping branching and if it is minimal for the order relation induced by \( \preceq \) on the quotient of \( Br(E, R) \) by the equivalence relation generated by: \( f_\bullet \sim g_\bullet \) if \( f_\bullet \preceq g_\bullet \) and \( g_\bullet \preceq f_\bullet \), and \( \sigma \cdot f_\bullet \sim f_\bullet \) for any \( \sigma \in \Sigma_n \).

Example 4.5. In the case where \( \mathcal{O} \) is the monoid operad, then we recover the usual notion of critical branching. If \( \mathcal{O} \) is the the operad of sets equipped with the action of a group \( G \), let us show that any non-aspherical branching is a critical branching.

First note that since all operations are unary there is no Peiffer branching, so all non-aspherical branchings are overlapping branchings. Let \( f_\bullet \) be an overlapping branching, and suppose there exists a branching \( g_\bullet \) such that \( f_\bullet \preceq g_\bullet \). Then there exists \( m \in \mathcal{O}(1) \) such that \( f = m(g) \). But since \( \mathcal{O}(1) \) is a group, we have \( m^{-1}(f) = g \) and therefore \( f \preceq g \). Since this is true for any \( g, f \) is minimal. It is therefore a critical branching.

Proposition 4.6. Let \( \mathcal{O} \) be an operad, and suppose that \( \mathcal{O}(1) \) is a group. Then for any \( \mathcal{O} \)-rewriting system, the critical branchings generate all the overlapping branchings. In other words for any overlapping branching \( f_\bullet \), there exists a critical branching \( g_\bullet \) such that \( f_\bullet \preceq g_\bullet \).

Proof. Let \( (E, R) \) be an \( \mathcal{O} \)-rewriting system. An element \( u \in E^+ \) is the data of an element \( m \in \mathcal{O}_n \) and elements \( x_1, \ldots, x_n \in E \). We call \( n \) the length of \( u \). If \( f_\bullet \) is a local \( n \)-branching, we denote by \( l(f_\bullet) \) the length of the source of \( f_\bullet \). Notice that we have for all local branchings:

\[
f_\bullet \preceq g_\bullet \implies l(f_\bullet) \preceq l(g_\bullet)
\]
Moreover since $O(1)$ is a group, then if $f_* \leq g_*$ and $I(f_*) = I(g_*)$ we have $f_* \sim g_*$ (see Example 4.5).

Let now $f_*$ be an overlapping branching. There exists $g_*$ of length minimal such that $g_* \leq f_*$. Let us prove that $g_*$ is a critical branching. Indeed if $h_* \leq g_*$ then $h_* \leq f_*$. Since $g_*$ is of minimal length among such branchings, $I(g_*) = I(h_*)$. So finally $g_* \sim h_*$. Since this is true for any overlapping branching smaller than $g_*$, $g_*$ is a critical branching.

Example 4.7. Let us now give an example of an operad $O$ and an $O$-rewriting system $(E,R)$ such that the critical branchings do not generate all the overlapping branchings.

Let $M$ be the monoid generated by $p,q$, and $a_n, b_n$ and $c_n$, for all $n \geq 0$, under the relations, for all $n \geq 0$:

\[
a_n p = b_n q \quad a_{n+1} = c_n a_n \quad b_{n+1} = c_n b_n
\]

Let $O$ be the operad of sets equipped with an action of $M$. In particular since $M$ is not a group we cannot apply Proposition 4.6. Let $E = \{x, y\}$ and $R = \{f, g\}$ with $f : p(x) \to y$ and $g : q(x) \to y$.

A superposition 2-branching is necessarily of the form $(m_1(f), m_2(g))$ for some $m_1, m_2 \in M$ such that $m_1 p = m_2 q$. Using the presentation of $M$, necessarily $m_1 = m'a_n$ and $m_2 = m'b_n$. In all cases we have $(m_1(f), m_2(g)) \geq (a_n(f), b_n(g))$.

Let us now show that none of the branchings of this form is minimal. Indeed we have for all $n \geq 0$:

\[
(a_n(f), b_n(g)) = (c_n a_{n+1}(f), c_n b_{n+1}(g)) = c_n(a_{n+1}(f), b_{n+1}(g)) > (a_{n+1}(f), b_{n+1}(g))
\]

References


